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Quadratic approximation of solutions for boundary value problems with nonlocal boundary conditions[☆]

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ABSTRACT

In this paper, using the quasilinearization method coupled with the method of upper and lower solutions, we study a class of second-order nonlinear boundary value problems with nonlocal boundary conditions. We establish some sufficient conditions under which corresponding monotone sequences converge uniformly and quadratically to the unique solution of the problem. An example is also included to illustrate the main result.

1. Introduction

It is well known that the quasilinearization method (QLM) of Bellman and Kalaba [1,2] was developed with the aim of solving a nonlinear ordinary or partial differential equation as a limit of a sequence of linear differential equations. This goal is easily understandable since there is no useful technique for obtaining the general solution of a nonlinear equation in terms of a finite set of particular solutions, in contrast to the case for linear equations, which can often be solved analytically or numerically in a convenient fashion using superposition. In addition, the QLM sequence can be constructed to assure quadratic convergence and, if possible, monotonicity. Recently, the method was generalized and extended using less restrictive assumptions so as to apply to a large class of differential problems; for details see [3–23].

In this paper, we shall consider the following boundary value problem:

$$\begin{cases} x''=f(t,x), \quad t\in I=[0,1],\\ x(0)-g_1(x'(0))=\displaystyle\int_0^1 h_1(x(s))\mathrm{d}s,\\ x(1)+g_2(x'(1))=\displaystyle\int_0^1 h_2(x(s))\mathrm{d}s \end{cases} \tag{1.1}$$

where $f: I\times\mathbb{R}\to\mathbb{R}$ and $g_i, h_i:\mathbb{R}\to\mathbb{R}$ are continuous, $i=1,2$.

The purpose of this paper is to continue developing the recent ideas regarding problems of type (1.1). Concretely, we apply the quasilinearization method coupled with the method of upper and lower solutions to obtain approximate solutions

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to nonlinear BVP (1.1) assuming some appropriate properties for f, g_i and $h_i (i = 1, 2)$. Then, we can show that some monotone sequences converge monotonically and quadratically in some norm to the unique solution of BVP (1.1) in the closed set generated by lower and upper solutions. As far as we know, this problem—where our boundary conditions include nonlinear derivative terms—has not been studied in the available reference materials. Because of our nonlinear and nonlocal boundary conditions, we generalize and extend some existing results. Boundary value problems with nonlinear boundary conditions have been studied by some authors, for example [9–15] and the references therein. For boundary value problems with nonlocal boundary conditions and comments on their importance, we refer the reader to the papers [16–18] and the references therein.

It is worth pointing out that [10] studied a class of boundary value problems with the following boundary conditions:

$$\begin{cases} g(x(a), x(b), px'(a)) = 0, \\ h(x(a), x(b), px'(b)) = 0. \end{cases}$$

The authors presented a generalized quasilinearization method of the problem under a very smart assumption (see Theorem 5 of [10]). Moreover, in [11], the same authors showed a generalization of [10] to a class of singular problems.

This paper contains two sections besides the introductory one. In Section 2, we give some basic concepts and some preparatory theorems. Then we present and prove the main result concerning the quasilinearization method. This is the content of Section 3.

2. Preliminaries

In this section, we will present some basic concepts and some preparatory results for later use.

Lemma 2.1. *Consider the following boundary value problem:*

$$\begin{cases} x'' = \sigma(t), \quad t \in [0, 1], \\ x(0) - g_1(x'(0)) = \int_0^1 \rho_1(s)\mathrm{d}s, \\ x(1) + g_2(x'(1)) = \int_0^1 \rho_2(s)\mathrm{d}s. \end{cases} \tag{2.1}$$

Assume that:

(1) $\sigma, \rho_i \in C[0, 1]$ $(i = 1, 2)$;
(2) $g_i \in C^1(\mathbb{R}), g_i(s) \to +\infty$ *if* $s \to +\infty, g_i(s) \to -\infty$ *if* $s \to -\infty, g_i'(s) > 0, i = 1, 2, s \in \mathbb{R}$.

Then BVP (2.1) has a unique solution in the segment $[0, 1]$.

Proof. It is easy to see that a solution of BVP (2.1) is

$$x(t) = c_1 + c_2 t + \varphi(t),$$

where $\varphi(t) \equiv \int_0^t \int_0^s \sigma(v)\mathrm{d}v\mathrm{d}s$, and (c_1, c_2) is determined by

$$\begin{cases} c_1 - g_1(c_2) = \int_0^1 \rho_1(s)\mathrm{d}s, \\ c_1 + c_2 + \varphi(1) + g_2(c_2 + \varphi'(1)) = \int_0^1 \rho_2(s)\mathrm{d}s. \end{cases}$$

From the assumptions and using standard arguments, we may see that (c_1, c_2) exists uniquely. In fact, from the last two equations, we have

$$c_2 + g_1(c_2) + g_2(c_2 + \varphi'(1)) = \int_0^1 \rho_2(s)\mathrm{d}s - \int_0^1 \rho_1(s)\mathrm{d}s - \varphi(1).$$

Noticing the assumptions, especially the strict monotonicity of the function

$$c_2 + g_1(c_2) + g_2(c_2 + \varphi'(1))$$

w.r.t. c_2, we obtain the existence and uniqueness of c_2. And then, it follows that c_1 exists uniquely.

The proof is complete. $\square$

Remark 2.1. Under the assumptions of Lemma 2.1,

$$\text{BVP (2.1)} \iff \begin{cases} x'' = \sigma(t), \quad t \in [0,1], \\ x(0) - g_1'(x'(0)) \cdot x'(0) = g_1(x'(0)) - g_1'(x'(0)) \cdot x'(0) + \displaystyle\int_0^1 \rho_1(s)\mathrm{d}s, \\ x(1) + g_2'(x'(1)) \cdot x'(1) = g_2'(x(1)) \cdot x'(1) - g_2(x'(1)) + \displaystyle\int_0^1 \rho_2(s)\mathrm{d}s \end{cases}$$

$$\iff x(t) = P(t) + \int_0^1 G(t,s)\sigma(s)\mathrm{d}s,$$

where

$$P(t) = \frac{1}{\Delta}\left[\left(1 - t + g_2'(x'(1))\right)\left(g_1(x'(0)) - g_1'(x'(0)) \cdot x'(0) + \int_0^1 \rho_1(s)\mathrm{d}s\right)\right.$$
$$\left. + \left(t + g_1'(x'(0))\right)\left(g_2'(x(1)) \cdot x'(1) - g_2(x'(1)) + \int_0^1 \rho_2(s)\mathrm{d}s\right)\right]$$

is the unique solution of the problem

$$\begin{cases} y'' = 0, \quad t \in [0,1], \\ y(0) - g_1'(x'(0)) \cdot y'(0) = g_1(x'(0)) - g_1'(x'(0)) \cdot x'(0) + \displaystyle\int_0^1 \rho_1(s)\mathrm{d}s, \\ y(1) + g_2'(x'(1)) \cdot y'(1) = g_2'(x(1)) \cdot x'(1) - g_2(x'(1)) + \displaystyle\int_0^1 \rho_2(s)\mathrm{d}s, \end{cases}$$

and

$$G(t,s) = \begin{cases} -\dfrac{1}{\Delta}\left(g_1'(x'(0)) + t\right)\left(1 + g_2'(x'(1)) - s\right), & 0 \leqslant t < s \leqslant 1; \\ -\dfrac{1}{\Delta}\left(g_1'(x'(0)) + s\right)\left(1 + g_2'(x'(1)) - t\right), & 0 \leqslant s < t \leqslant 1 \end{cases}$$

is the Green's function of the problem

$$\begin{cases} y'' = 0, \quad t \in [0,1], \\ y(0) - g_1'(x'(0)) \cdot y'(0) = 0, \\ y(1) + g_2'(x'(1)) \cdot y'(1) = 0, \end{cases}$$

where

$$\Delta = \begin{vmatrix} 1 & -g_1'(x'(0)) \\ 1 & 1 + g_2'(x'(1)) \end{vmatrix}.$$

We note that $G(t,s) < 0$ on $(0,1) \times (0,1)$.

Definition 2.1. Let $\alpha, \beta \in C^2[0,1]$. The function α is called a lower solution of BVP (1.1) if

$$\begin{cases} \alpha''(t) \geqslant f(t, \alpha(t)), \quad t \in I = [0,1], \\ \alpha(0) - g_1(\alpha'(0)) \leqslant \displaystyle\int_0^1 h_1(\alpha(s))\mathrm{d}s, \\ \alpha(1) + g_2(\alpha'(1)) \leqslant \displaystyle\int_0^1 h_2(\alpha(s))\mathrm{d}s. \end{cases}$$

Similarly, β is called an upper solution of the BVP (1.1), if β satisfies similar inequalities in the reverse direction.

Now, we state and prove the existence and uniqueness of solutions in an ordered interval generated by the lower and upper solutions of the boundary value problem (1.1).

Theorem 2.1. *Assume that:*

(1) $\alpha, \beta \in C^2[0,1]$ *are lower and upper solutions of BVP* (1.1), *respectively, such that* $\alpha(t) \leqslant \beta(t),\ t \in [0,1]$;
(2) $g_i \in C^1(\mathbb{R}), g_i'(s) \geqslant 0, i = 1, 2, s \in \mathbb{R}$;
(3) $h_i \in C^1(\mathbb{R}), h_i'(s) \geqslant 0, i = 1, 2, s \in \mathbb{R}$.

Then there exists a solution $x \in C^2[0, 1]$ *of BVP* (1.1) *such that*

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \quad t \in [0, 1].$$

Proof. Define

$$\tilde{x} = \delta(\alpha, x, \beta) = \begin{cases} \alpha, & x < \alpha, \\ x, & x \in [\alpha, \beta], \\ \beta, & x > \beta. \end{cases}$$

Consider the following modified problem:

$$\begin{cases} x'' = F(t, x) \equiv F^*(t), \\ x(0) = \delta\left(\alpha(0), g_1(x'(0)) + \displaystyle\int_0^1 h_1(x(s))\mathrm{d}s, \beta(0)\right), \\ x(1) = \delta\left(\alpha(1), -g_2(x'(1)) + \displaystyle\int_0^1 h_2(x(s))\mathrm{d}s, \beta(1)\right), \end{cases} \tag{2.2}$$

where

$$F(t, x) = f(t, \tilde{x}) + h(x),$$

$$h(x) = \begin{cases} \dfrac{x - \beta}{1 + |x - \beta|}, & x > \beta, \\ 0, & x \in [\alpha, \beta], \\ \dfrac{x - \alpha}{1 + |x - \alpha|}, & x < \alpha. \end{cases}$$

Notice that BVP (2.2) may be rewritten as an integral equation. Since F^* is continuous and bounded, employing the standard arguments (cf. for example [24]), by using of the theory of topological degree we may get that the integral equation has at least one solution $x \in C^2[0, 1]$ on the set

$$\Omega = \{x(t) : \|x^{(i)}\| < K, i = 0, 1, K \text{ is some sufficientlly large constant}, \forall t \in [0, 1]\},$$

where $\|\cdot\|$ is the usual maximum norm.

We now argue that each solution $x(t)$ of BVP (2.2) satisfies $\alpha(t) \leqslant x(t) \leqslant \beta(t), \forall t \in [0, 1]$. We shall show that $\alpha(t) \leqslant x(t), \forall t \in [0, 1]$. Define $R(t) \equiv \alpha(t) - x(t), t \in [0, 1]$. Assume, for the sake of contradiction, that there exists some $t_0 \in [0, 1]$ such that

$$R(t_0) = \max_{t \in [0,1]} R(t) = \max_{t \in [0,1]} (\alpha(t) - x(t)) > 0.$$

From the boundary conditions of BVP (2.2), it is easy to see that $t_0 \neq 0, 1$. Thus, we may suppose that $t_0 \in (0, 1)$. Then $R(t_0) > 0, R'(t_0) = 0, R''(t_0) \leqslant 0$. Hence

$$\begin{aligned} 0 \geqslant R''(t_0) &= \alpha''(t_0) - x''(t_0) \\ &\geqslant f(t_0, \alpha(t_0)) - F(t_0, x(t_0)) \\ &= f(t_0, \alpha(t_0)) - [f(t_0, \tilde{x}(t_0)) + h(x(t_0))] \\ &= -h(x(t_0)) > 0, \end{aligned}$$

which is a contradiction. Therefore, $x(t) \geqslant \alpha(t)$ holds. A similar proof shows that $x(t) \leqslant \beta(t)$.

Now, we prove that the solution $x \in [\alpha, \beta]$ of BVP (2.2) is a solution of BVP (1.1). In fact, it is enough to prove that the solution x also satisfies the boundary conditions of BVP (1.1). By using of the method of classification, a proof similar to [23, Theorem 3.1] can show this argument. Here, for completeness, we show the details. We only prove that the solution x of BVP (2.2) satisfies the following boundary condition:

$$x(0) - g_1(x'(0)) = \int_0^1 h_1(x(s))\mathrm{d}s$$

and we consider the following three cases:

Case 1: Suppose that

$$\alpha(0) \leqslant g_1(x'(0)) + \int_0^1 h_1(x(s))\mathrm{d}s \leqslant \beta(0).$$

Then by the definition of the function δ, one has

$$x(0) = g_1(x'(0)) + \int_0^1 h_1(x(s))\mathrm{d}s.$$

That is,

$$x(0) - g_1(x'(0)) = \int_0^1 h_1(x(s))\mathrm{d}s.$$

Case 2: Suppose that

$$\alpha(0) > g_1(x'(0)) + \int_0^1 h_1(x(s))\mathrm{d}s.$$

Then by the definition of the function δ, one has

$$x(0) = \alpha(0).$$

Thus, $x(s) \geqslant \alpha(s)$ implies that $x'(0) \geqslant \alpha'(0)$. Noticing the monotonicity of g_1 and h_1,

$$\alpha(0) > g_1(x'(0)) + \int_0^1 h_1(x(s))\mathrm{d}s \geqslant g_1(\alpha'(0)) + \int_0^1 h_1(\alpha(s))\mathrm{d}s,$$

which contradicts Definition 2.1. Therefore, this case cannot hold.
Case 3: Suppose that

$$\beta(0) < g_1(x'(0)) + \int_0^1 h_1(x(s))\mathrm{d}s.$$

Then by the definition of the function δ, one has

$$x(0) = \beta(0).$$

By a similar argument, we can deduce that this case cannot hold.

To sum up, we get that the solution x of BVP (2.2) satisfies the following boundary condition:

$$x(0) - g_1(x'(0)) = \int_0^1 h_1(x(s))\mathrm{d}s.$$

This completes the proof of the theorem. □

Theorem 2.2. *Assume that:*

(1) $\alpha, \beta \in C^2[0, 1]$ *are lower and upper solutions of BVP* (1.1)*, respectively;*
(2) $f(t, x) \in C^1([0, 1] \times \mathbb{R}), f_x(t, x) > 0, (t, x) \in [0, 1] \times \mathbb{R}$;
(3) $g_i \in C^1(\mathbb{R}), 0 \leqslant g_i'(s), i = 1, 2, s \in \mathbb{R}$;
(4) $h_i \in C^1(\mathbb{R}), 0 \leqslant h_i'(s) < 1, i = 1, 2, s \in \mathbb{R}$.

Then $\alpha(t) \leqslant \beta(t), t \in [0, 1]$.

Proof. Define $S(t) \equiv \alpha(t) - \beta(t), t \in [0, 1]$. As in the proof of Theorem 2.1, assume for the sake of contradiction that there exists some $t_0 \in [0, 1]$ such that

$$S(t_0) = \max_{t\in[0,1]} S(t) = \max_{t\in[0,1]} (\alpha(t) - \beta(t)) > 0.$$

Case 1: Suppose that $t_0 \in (0, 1)$. Then $S(t_0) > 0, S'(t_0) = 0, S''(t_0) \leqslant 0$. Hence

$$\begin{aligned} 0 \geqslant S''(t_0) &= \alpha''(t_0) - \beta''(t_0) \\ &\geqslant f(t_0, \alpha(t_0)) - f(t_0, \beta(t_0)) > 0, \end{aligned}$$

a contradiction.
Case 2: Suppose that $t_0 = 0$. Then $S(0) > 0, S'(0) \leqslant 0$. Hence

$$\begin{aligned} S(0) = \alpha(0) - \beta(0) &\leqslant \big(\alpha(0) - g_1(\alpha'(0))\big) - \big(\beta(0) - g_1(\beta'(0))\big) \\ &\leqslant \int_0^1 h_1(\alpha(s))\mathrm{d}s - \int_0^1 h_1(\beta(s))\mathrm{d}s \\ &= \int_0^1 h_1'(\eta(s))(\alpha(s) - \beta(s))\mathrm{d}s \\ &\leqslant \int_0^1 h_1'(\eta(s))S(0)\mathrm{d}s < S(0), \end{aligned}$$

where η is between α and β. Thus, we get a contradiction.

Case 3: Suppose that $t_0 = 1$. Then $S(1) > 0$, $S'(1) \geqslant 0$. A similar proof shows that this case cannot hold.
To sum up, $\alpha(t) \leqslant \beta(t)$, $t \in [0, 1]$. □

Corollary 2.1. *Assume that:*

(1) $\alpha, \beta \in C^2[0, 1]$ *are lower and upper solutions of BVP* (1.1), *respectively;*
(2) $f(t, x) \in C^1([0, 1] \times \mathbb{R})$, $f_x(t, x) > 0$, $(t, x) \in [0, 1] \times \mathbb{R}$;
(3) $g_i \in C^1(\mathbb{R})$, $g_i'(s) \geqslant 0$, $i = 1, 2$, $s \in \mathbb{R}$;
(4) $h_i \in C^1(\mathbb{R})$, $0 \leqslant h_i'(s) < 1$, $i = 1, 2$, $s \in \mathbb{R}$.

Then BVP (1.1) *has a unique solution* $x \in C^2[0, 1]$ *of BVP* (1.1) *such that*

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \quad t \in [0, 1].$$

3. Main result

Now, we present and prove our main result:

Theorem 3.1. *Assume that:*

(1) $\alpha, \beta \in C^2[0, 1]$ *are lower and upper solutions of BVP* (1.1), *respectively;*
(2) $f(t, x) \in C^1([0, 1] \times \mathbb{R})$, $f_x(t, x) > 0$, $(t, x) \in [0, 1] \times \mathbb{R}$;
(3) $g_i \in C^2(\mathbb{R})$, $g_i'(s) > 0$, $i = 1, 2$, $g_1''(s) \geqslant 0$, $g_2''(s) \leqslant 0$, $s \in \mathbb{R}$;
(4) $h_i \in C^2(\mathbb{R})$, $0 \leqslant h_i'(s) < \lambda_i < 1$, $h_i''(s) \geqslant 0$, $i = 1, 2$, $s \in \mathbb{R}$.

Then, there exists a monotone sequence $\{\alpha_n\}$ *which converges uniformly to the unique solution* x *of BVP* (1.1) *and the convergence is quadratic in the* C^1 *norm.*

Proof. In view of the assumptions, by Corollary 2.1, BVP (1.1) has a unique solution $x \in C^2[0, 1]$ such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \quad t \in [0, 1].$$

We divide the proof into three steps.

Step 1. Define some auxiliary functions
Set

$$\Phi(t, x) \equiv F(t, x) - f(t, x) \quad \text{on } [0, 1] \times \mathbb{R},$$

where $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is such that $F(t, x)$, $F_x(t, x)$, $F_{xx}(t, x)$ are continuous on $[0, 1] \times \mathbb{R}$ and

$$F_{xx}(t, x) \leqslant 0, \quad (t, x) \in [0, 1] \times \mathbb{R}.$$

Using the mean value theorem and the assumptions, we obtain, whether or not $x \geqslant y$,

$$\begin{aligned}
&f(t, x) \leqslant f(t, y) + F_x(t, y)(x - y) - [\Phi(t, x) - \Phi(t, y)] \equiv \overline{F}(t, x; y),\\
&g_1(x) \geqslant g_1(y) + g_1'(y)(x - y) \equiv \overline{G}_1(x; y),\\
&g_2(x) \leqslant g_2(y) + g_2'(y)(x - y) \equiv \overline{G}_2(x; y),\\
&h_i(x) \geqslant h_i(y) + h_i'(y)(x - y) \equiv \overline{H}_i(x; y)
\end{aligned}$$

for any $(t, x, y) \in [0, 1] \times \mathbb{R}^2$, $i = 1, 2$. In particular, we consider the proof only on the set $\Omega = \{(t, x) : t \in [0, 1], x \in [\alpha, \beta]\}$.

Step 2. Construct the convergent sequence
Now, set $\alpha_0 = \alpha$ and consider the following BVP:

$$\begin{cases}
x'' = \overline{F}(t, x; \alpha_0(t)),\\
x(0) - \overline{G}_1(x'(0); \alpha_0'(0)) = \displaystyle\int_0^1 \overline{H}_1(x(s); \alpha_0(s))\mathrm{d}s,\\
x(1) + \overline{G}_2(x'(1); \alpha_0'(1)) = \displaystyle\int_0^1 \overline{H}_2(x(s); \alpha_0(s))\mathrm{d}s.
\end{cases} \tag{3.1}$$

Then

$$\begin{aligned}
&\alpha_0''(t) \geqslant f(t,\alpha_0(t)) = \overline{F}(t,\alpha_0(t);\alpha_0(t)),\\
&\alpha(0) - \overline{G}_1(\alpha_0'(0);\alpha_0'(0)) \;=\; \alpha(0) - g_1(\alpha_0'(0))\\
&\qquad\qquad\leqslant \int_0^1 h_1(\alpha_0(s))\mathrm{d}s = \int_0^1 \overline{H}_1(\alpha_0(s);\alpha_0(s))\mathrm{d}s,\\
&\alpha_0(1) + \overline{G}_2(\alpha_0'(1);\alpha_0'(1)) \;=\; \alpha_0(1) + g_2(\alpha_0'(1))\\
&\qquad\qquad\leqslant \int_0^1 h_2(\alpha_0(s))\mathrm{d}s = \int_0^1 \overline{H}_2(\alpha_0(s);\alpha_0(s))\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\beta''(t) \leqslant f(t,\beta(t)) \leqslant \overline{F}(t,\beta(t);\alpha_0(t)),\\
&\beta(0) - \overline{G}_1(\beta'(0);\alpha_0'(0)) \;\geqslant\; \beta(0) - g_1(\beta'(0))\\
&\qquad\qquad\geqslant \int_0^1 h_1(\beta(s))\mathrm{d}s \geqslant \int_0^1 \overline{H}_1(\beta(s);\alpha_0(s))\mathrm{d}s,\\
&\beta(1) + \overline{G}_2(\beta'(1);\alpha_0'(1)) \;\geqslant\; \beta(1) + g_2(\beta'(1))\\
&\qquad\qquad\geqslant \int_0^1 h_2(\beta(s))\mathrm{d}s \geqslant \int_0^1 \overline{H}_2(\beta(s);\alpha_0(s))\mathrm{d}s,
\end{aligned}$$

which implies that α_0 and β are lower and upper solutions of BVP (3.1), respectively. Also, it is easy to see that $\overline{F}$, $\overline{G}_i$ and $\overline{H}_i$ $(i = 1, 2)$ are such that the assumptions of Corollary 2.1 hold. Hence, by Corollary 2.1, BVP (3.1) has a unique solution $\alpha_1 \in C^2[0, 1]$ such that

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta(t), \quad t \in [0, 1].$$

Furthermore, we note that

$$\begin{aligned}
&\alpha_1''(t) = \overline{F}(t,\alpha_1(t);\alpha_0(t)) \geqslant f(t,\alpha_1(t)),\\
&\alpha_1(0) - g_1(\alpha_1'(0)) \;\leqslant\; \alpha_1(0) - \overline{G}_1(\alpha_1'(0);\alpha_0'(0))\\
&\qquad\qquad= \int_0^1 \overline{H}_1(\alpha_1(s);\alpha_0(s))\mathrm{d}s \leqslant \int_0^1 h_1(\alpha_1(s))\mathrm{d}s,\\
&\alpha_1(1) + g_2(\alpha_1'(1)) \;\leqslant\; \alpha_1(1) + \overline{G}_2(\alpha_1'(1);\alpha_0'(1))\\
&\qquad\qquad= \int_0^1 \overline{H}_2(\alpha_1(s);\alpha_0(s))\mathrm{d}s \leqslant \int_0^1 h_2(\alpha_1(s))\mathrm{d}s
\end{aligned}$$

which implies that α_1 is a lower solution of BVP (1.1).

Now, consider the following BVP:

$$\begin{cases}
x'' = \overline{F}(t,x;\alpha_1(t)),\\
x(0) - \overline{G}_1(x'(0);\alpha_1'(0)) = \int_0^1 \overline{H}_1(x(s);\alpha_1(s))\mathrm{d}s,\\
x(1) + \overline{G}_2(x'(1);\alpha_1'(1)) = \int_0^1 \overline{H}_2(x(s);\alpha_1(s))\mathrm{d}s.
\end{cases} \tag{3.2}$$

Again, we find that α_1 and β are lower and upper solutions of BVP (3.2), respectively. Also, it is easy to see that $\overline{F}$, $\overline{G}_i$ and $\overline{H}_i$ $(i = 1, 2)$ are such that the assumptions of Corollary 2.1 hold. Hence, by Corollary 2.1, BVP (3.2) has a unique solution $\alpha_2 \in C^2[0, 1]$, such that

$$\alpha_1(t) \leqslant \alpha_2(t) \leqslant \beta(t), \quad t \in [0, 1].$$

Employing the same arguments successively, we conclude that for all n and $t \in [0, 1]$,

$$\alpha = \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n \leqslant \beta,$$

where the elements of the monotone sequence $\{\alpha_n\}$ are the unique solutions of the BVP

$$\begin{cases}
x'' = \overline{F}(t,x;\alpha_{n-1}),\\
x(0) - \overline{G}_1(x'(0);\alpha_{n-1}'(0)) = \int_0^1 \overline{H}_1(x(s);\alpha_{n-1}(s))\mathrm{d}s,\\
x(1) + \overline{G}_2(x'(1);\alpha_{n-1}'(1)) = \int_0^1 \overline{H}_2(x(s);\alpha_{n-1}(s))\mathrm{d}s.
\end{cases}$$

Consider the following Robin type BVP:

$$\begin{cases} x'' = \overline{F}(t, \alpha_n; \alpha_{n-1}), \\ x(0) - \overline{G}_1(x'(0); \alpha'_{n-1}(0)) = \int_0^1 \overline{H}_1(\alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s, \\ x(1) + \overline{G}_2(x'(1); \alpha'_{n-1}(1)) = \int_0^1 \overline{H}_2(\alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s. \end{cases} \tag{3.3}$$

From Lemma 2.1, BVP (3.3) has a unique solution. It is easy to see that α_n is the unique solution. Thus, we may conclude that

$$\alpha_n(t) = \overline{P}(t) + \int_0^1 \overline{G}(t, s)\overline{F}(s, \alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s, \tag{3.4}$$

where

$$\overline{P}(t) = \frac{1}{\Delta}\left[\left(1 - t + g_2'(\alpha'_{n-1}(1))\right)\left(g_1(\alpha'_{n-1}(0)) - g_1'(\alpha'_{n-1}(0)) \cdot \alpha'_{n-1}(0) + \int_0^1 \overline{H}_1(\alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s\right)\right.$$
$$\left. + \left(t + g_1'(\alpha'_{n-1}(0))\right)\left(g_2'(\alpha_{n-1}(1)) \cdot \alpha'_{n-1}(1) - g_2(\alpha'_{n-1}(1)) + \int_0^1 \overline{H}_2(\alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s\right)\right]$$

and

$$\overline{G}(t, s) = \begin{cases} -\frac{1}{\Delta}\left(g_1'(\alpha'_{n-1}(0)) + t\right)\left(1 + g_2'(\alpha'_{n-1}(1)) - s\right), & 0 \leqslant t < s \leqslant 1; \\ -\frac{1}{\Delta}\left(g_1'(\alpha'_{n-1}(0)) + s\right)\left(1 + g_2'(\alpha'_{n-1}(1)) - t\right), & 0 \leqslant s < t \leqslant 1 \end{cases}$$

with

$$\Delta = \begin{vmatrix} 1 & -g_1'(\alpha'_{n-1}(0)) \\ 1 & 1 + g_2'(\alpha'_{n-1}(1)) \end{vmatrix}.$$

Employing the fact that [0, 1] is compact and the monotone convergence is pointwise, it follows that the convergence of the sequence is uniform. If $x(t)$ is the limit point of the sequence $\alpha_n(t)$, then passing to the limit $n \to \infty$, (3.4) gives

$$x(t) = P(t) + \int_0^1 G(t, s)f(s, x(s))\mathrm{d}s.$$

Thus, by Remark 2.1, $x(t)$ is the solution of the BVP (1.1).

Step 3. Show the quadratic convergence

First, we define a norm $\|\cdot\|$ as follows:

$$|||u||| \triangleq \max\{\|u\|, \|u'\|\}.$$

To show the quadratic rate of convergence in the sense of the norm $|||\cdot|||$, define the error function

$$e_n(t) \equiv x(t) - \alpha_n(t) \geqslant 0, \quad t \in [0, 1].$$

Then

$$\begin{aligned} e_n''(t) &= x''(t) - \alpha_n''(t) \\ &= f(t, x(t)) - f(t, \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))(\alpha_n(t) - \alpha_{n-1}(t)) + [(\Phi(t, \alpha_n(t)) - \Phi(t, \alpha_{n-1}(t))] \\ &= F(t, x(t)) - F(t, \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))(\alpha_n(t) - \alpha_{n-1}(t)) + [\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\ &= F_x(t, \xi_1)(x(t) - \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))(\alpha_n(t) - \alpha_{n-1}(t)) + [\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\ &= (F_x(t, \xi_1) - F_x(t, \alpha_{n-1}(t)))(x(t) - \alpha_{n-1}(t)) + F_x(t, \alpha_{n-1}(t))(x(t) - \alpha_n(t)) + [\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\ &= F_{xx}(t, \xi_2)(\xi_1 - \alpha_{n-1})(x(t) - \alpha_{n-1}(t)) + F_x(t, \alpha_{n-1}(t))(x(t) - \alpha_n(t)) - \Phi_x(t, \xi_3)(x(t)) - \alpha_n(t)) \\ &= F_{xx}(t, \xi_2)(\xi_1 - \alpha_{n-1})(x(t) - \alpha_{n-1}(t)) + [F_x(t, \alpha_{n-1}(t)) - \Phi_x(t, \xi_3)](x(t) - \alpha_n(t)), \end{aligned}$$

where $\alpha_{n-1}(t) \leqslant \xi_1 \leqslant \xi_2 \leqslant x(t)$ and $\alpha_n(t) \leqslant \xi_3 \leqslant x(t)$. Since $F_{xx} \leqslant 0$, noticing that the set Ω is compact, it follows by assumption 2 that there exists $\gamma > 0$ and an integer N such that

$$F_x(t, \alpha_{n-1}(t)) - \Phi_x(t, \xi_3) \geqslant \gamma, \quad t \in [0, 1], n \geqslant N.$$

Hence, we obtain

$$e_n''(t) \geqslant \gamma e_n(t) - M\|e_{n-1}\|^2, \tag{3.5}$$

where $M \geqslant |F_{xx}(t,s)|$, for $s \in [\alpha_{n-1}(t), x(t)]$, $t \in [0,1]$. Furthermore,

$$\begin{aligned}
e_n(0) &= x(0) - \alpha_n(0) \\
&= \left(g_1(x'(0)) + \int_0^1 h_1(x(s))\mathrm{d}s\right) - \left(\overline{G}_1(\alpha_n'(0); \alpha_{n-1}'(0)) + \int_0^1 \overline{H}_1(\alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s\right) \\
&= \left(g_1(x'(0)) - \overline{G}_1(\alpha_n'(0); \alpha_{n-1}'(0))\right) + \left(\int_0^1 h_1(x(s))\mathrm{d}s - \int_0^1 \overline{H}_1(\alpha_n(s); \alpha_{n-1}(s))\mathrm{d}s\right) \\
&= g_1(x'(0)) - g_1(\alpha_{n-1}'(0)) - g_1'(\alpha_{n-1}'(0))(\alpha_n'(0) - \alpha_{n-1}'(0)) + \int_0^1 \left[h_1(x(s)) - \overline{H}_1(\alpha_n(s); \alpha_{n-1}(s))\right]\mathrm{d}s \\
&= g_1'(\alpha_{n-1}'(0))(x'(0) - \alpha_{n-1}'(0)) + \frac{g_1''(\xi_4)}{2}(x'(0) - \alpha_{n-1}'(0))^2 \\
&\quad - g_1'(\alpha_{n-1}'(0))(\alpha_n'(0) - x'(0)) - g_1'(\alpha_{n-1}'(0))(x'(0) - \alpha_{n-1}'(0)) + \int_0^1 \left[h_1(x(s)) - \overline{H}_1(\alpha_n(s); \alpha_{n-1}(s))\right]\mathrm{d}s \\
&= \frac{g_1''(\xi_4)}{2} e_{n-1}'^2(0) + g_1'(\alpha_{n-1}'(0))e_n'(0) + \int_0^1 \left[h_1'(\alpha_{n-1}(s))e_n(s) + \frac{h_1''(\xi_5)}{2} e_{n-1}^2(s)\right]\mathrm{d}s \\
&\leqslant \frac{g_1''(\xi_4)}{2} e_{n-1}'^2(0) + g_1'(\alpha_{n-1}'(0))e_n'(0) + \lambda_1 \int_0^1 e_n(s)\mathrm{d}s + \int_0^1 \frac{h_1''(\xi_5)}{2}\mathrm{d}s \|e_{n-1}\|^2,
\end{aligned}$$

where ξ_4 is between $\alpha_{n-1}'(0)$ and $x'(0)$, and $\alpha_{n-1}(s) \leqslant \xi_5 \leqslant x(s)$. Thus, recalling the definition of $|||\cdot|||$, we have

$$e_n(0) - g_1'(\alpha_{n-1}'(0))e_n'(0) \leqslant \lambda_1 \int_0^1 e_n(s)\mathrm{d}s + \frac{g_1''(\xi_4) + \int_0^1 h_1''(\xi_5)\mathrm{d}s}{2} |||e_{n-1}|||^2.$$

Similarly, we get

$$e_n(1) + g_2'(\alpha_{n-1}'(1))e_n'(1) \leqslant \lambda_2 \int_0^1 e_n(s)\mathrm{d}s + \frac{\int_0^1 h_2''(\xi_7)\mathrm{d}s - g_2''(\xi_6)}{2} |||e_{n-1}|||^2,$$

where ξ_6 is between $\alpha_{n-1}'(1)$ and $x'(1)$, and $\alpha_{n-1}(s) \leqslant \xi_7 \leqslant x(s)$. The boundedness of the functions ξ_5 and ξ_7, together with the condition $h_i \in C^2$ $(i = 1, 2)$, allows one to select two constants C_1, C_2 such that

$$C_1 \geqslant \frac{g_1''(\xi_4) + h_1''(\xi_5)}{2} \geqslant 0, \qquad C_2 \geqslant \frac{h_2''(\xi_7) - g_2''(\xi_6)}{2} \geqslant 0.$$

Then

$$\begin{aligned}
&e_n(0) - g_1'(\alpha_{n-1}'(0))e_n'(0) \leqslant \lambda \int_0^1 e_n(s)\mathrm{d}s + C|||e_{n-1}|||^2, \\
&e_n(1) + g_2'(\alpha_{n-1}'(1))e_n'(1) \leqslant \lambda \int_0^1 e_n(s)\mathrm{d}s + C|||e_{n-1}|||^2,
\end{aligned} \tag{3.6}$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$ and $C = \max\{C_1, C_2\}$. Now, we consider the following BVP:

$$\begin{cases}
y''(t) = \gamma y(t) - M\|e_{n-1}\|^2, \quad t \in [0,1], \\
y(0) - g_1'(\alpha_{n-1}'(0))y'(0) = \lambda \int_0^1 y(s)\mathrm{d}s + C|||e_{n-1}|||^2, \\
y(1) + g_2'(\alpha_{n-1}'(1))y'(1) = \lambda \int_0^1 y(s)\mathrm{d}s + C|||e_{n-1}|||^2.
\end{cases} \tag{3.7}$$

From (3.5) and (3.6), it follows that $e_n(t)$ is a lower solution of BVP (3.7). Choose $\gamma > 0$ small enough that $M(1-\lambda) \geqslant C\gamma$ and define

$$r(t) = \frac{M}{\gamma}\|e_{n-1}\|^2.$$

Then it is clear that

$$r''(t) = \gamma r(t) - M\|e_{n-1}\|^2 \equiv 0, \tag{3.8}$$

and

$$r(0) - g_1'(\alpha_{n-1}'(0))r'(0) \geqslant \lambda \int_0^1 r(s)\mathrm{d}s + C|||e_{n-1}|||^2,$$
$$r(1) + g_2'(\alpha_{n-1}'(1))r'(1) \geqslant \lambda \int_0^1 r(s)\mathrm{d}s + C|||e_{n-1}|||^2. \tag{3.9}$$

From (3.8) and (3.9), it follows that $r(t)$ is an upper solution of BVP (3.7). Hence, by Theorem 2.2, we obtain

$$e_n(t) \leqslant r(t) = \frac{M}{\gamma}\|e_{n-1}\|^2, \quad t \in [0, 1], n \geqslant N. \tag{3.10}$$

Next, we consider the case of $e_n'(t)$. From (3.5), it is easy to see that

$$e_n''(t) \geqslant -M\|e_{n-1}\|^2. \tag{3.11}$$

Integrating (3.11) from 0 to t, $t \in [0, 1]$, we have

$$\int_0^t e_n''(s)\mathrm{d}s \geqslant -M\|e_{n-1}\|^2,$$

or

$$e_n'(t) - e_n'(0) \geqslant -M\|e_{n-1}\|^2. \tag{3.12}$$

If $e_n'(0) \geqslant 0$, then (3.12) implies that

$$e_n'(t) \geqslant -M\|e_{n-1}\|^2. \tag{3.13}$$

If $e_n'(0) \leqslant 0$, then (3.6) implies that

$$\begin{aligned} -g_1'(\alpha_{n-1}'(0))e_n'(0) &\leqslant -e_n(0) + \lambda \int_0^1 e_n(s)\mathrm{d}s + C|||e_{n-1}|||^2 \\ &\leqslant \lambda \int_0^1 e_n(s)\mathrm{d}s + C|||e_{n-1}|||^2, \end{aligned}$$

or

$$e_n'(0) \geqslant -\frac{\lambda}{g_1'(\alpha_{n-1}'(0))}\int_0^1 e_n(s)\mathrm{d}s - \frac{C}{g_1'(\alpha_{n-1}'(0))}|||e_{n-1}|||^2.$$

Combining the last formula with formula (3.12), we have

$$e_n'(t) \geqslant -M\|e_{n-1}\|^2 - \frac{\lambda}{g_1'(\alpha_{n-1}'(0))}\int_0^1 e_n(s)\mathrm{d}s - \frac{C}{g_1'(\alpha_{n-1}'(0))}|||e_{n-1}|||^2. \tag{3.14}$$

From (3.13) and (3.14), we can obtain the lower bound of $e_n'(t)$. Similarly, we can obtain the upper bound of $e_n'(t)$. The details are as follows. Integrating (3.11) from t to 1, $t \in [0, 1]$, we have

$$\int_t^1 e_n''(s)\mathrm{d}s \geqslant -M\|e_{n-1}\|^2,$$

or

$$e_n'(t) - e_n'(1) \leqslant M\|e_{n-1}\|^2. \tag{3.15}$$

If $e_n'(1) \leqslant 0$, then (3.12) implies that

$$e_n'(t) \leqslant M\|e_{n-1}\|^2. \tag{3.16}$$

If $e_n'(1) \geqslant 0$, then (3.6) implies that

$$e_n'(1) \leqslant \frac{\lambda}{g_2'(\alpha_{n-1}'(1))}\int_0^1 e_n(s)\mathrm{d}s + \frac{C}{g_2'(\alpha_{n-1}'(1))}|||e_{n-1}|||^2.$$

Combining the last formula with formula (3.15), we have

$$e_n'(t) \leqslant M\|e_{n-1}\|^2 + \frac{\lambda}{g_2'(\alpha_{n-1}'(1))}\int_0^1 e_n(s)\mathrm{d}s + \frac{C}{g_2'(\alpha_{n-1}'(1))}|||e_{n-1}|||^2. \tag{3.17}$$

From (3.16) and (3.17), we can obtain the upper bound of $e_n'(t)$. Thus, from (3.13), (3.14), (3.16) and (3.17), it follows that there exist two positive constants M_1 and M_2 such that

$$|e_n'(t)| \leqslant M\|e_{n-1}\|^2 + M_1\int_0^1 e_n(s)\mathrm{d}s + M_2|||e_{n-1}|||^2.$$

In view of (3.10),

$$|e_n'(t)| \leqslant M\|e_{n-1}\|^2 + M_1\cdot\frac{M}{\gamma}\|e_{n-1}\|^2 + M_2|||e_{n-1}|||^2. \tag{3.18}$$

Formulae (3.10) and (3.18) imply that

$$|||e_n||| \leqslant M_3|||e_{n-1}|||^2,$$

where $M_3 = \max\{\frac{M}{\gamma}, M + M_1\cdot\frac{M}{\gamma} + M_2\}$. This establishes the quadratic convergence of the iterates. $\square$

Now we will illustrate the main result using the following example:

Example 3.1. Let

$$f(t,x) = \begin{cases} te^{x+1} + 2x, & \text{if } (t,x) \in [0,1]\times(-\infty,0),\\ \mathrm{e}t + x(\mathrm{e}t+2), & \text{if } (t,x) \in [0,1]\times[0,+\infty),\end{cases}$$

$$g(x) = \begin{cases} -x^4 + \frac{1}{2}x\sin x + 2x + \cos x, & \text{if } x\in(-\infty,0),\\ 2x+1, & \text{if } x\in[0,+\infty).\end{cases}$$

Consider the boundary value problem

$$\begin{cases} x'' = f(t,x), \quad t\in[0,1],\\ x(0) - k_1 g(x'(0)) = \displaystyle\int_0^1 \frac{cx(s)-1}{2}\mathrm{d}s,\\ x(1) + k_2 g(x'(1)) = \displaystyle\int_0^1 (cx(s)+1)\mathrm{d}s,\end{cases} \tag{3.19}$$

where $0 < k_1 \leqslant 1/12$, $1/6 \leqslant k_2 \leqslant 1$, $0 \leqslant c < 1$. It can easily be verified that $\alpha(t) = -1$ and $\beta(t) = t$ are the lower and upper solutions of BVP (3.19), respectively. Also the assumptions of Theorem 3.1 are satisfied. Hence we can obtain a monotone sequence of approximate solutions converging uniformly and quadratically to the unique solution of BVP (3.19).

Remark 3.1. From Theorem 3.1, we include or improve the results given in [1–23], since our system and boundary conditions are nonlinear and nonlocal; the nonlinearity of g_1, g_2 is especially notable. Obviously, not all of the results in the references are applicable to our example.

Last but not least, it should be pointed out that although this paper establishes some sufficient conditions under which corresponding monotone sequences converge uniformly and quadratically to the unique solution of the problem (1.1), the premise is that the lower and upper solutions are assumed to exist. It is well known that the problem of establishing how to get a pair of lower and upper solutions for a given BVP is a very difficult one in the theory of upper–lower solutions and remains unsolved.

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